HYDRAULICS OF RIVER FLOW UNDER ARCH BRIDGES — A PROGRESS REPORT

MAY 1959

NO. 12

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Technical Paper

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-A PROGRESS REPORT-

TO: K. B. Woods, Director
Joint Highway Research Project

May 14, 1959

FROM: H. L. Michael, Assistant Director
Joint Highway Research Project

File: 9-8-2
Project: C-36-62B

Attached is a technical paper entitled, "Hydraulics of River Flow Under Arch Bridges--A Progress Report," which has been prepared by Messrs. H. J. Owen, A. Sooky, S. T. Husain, and Prof. J. W. Delleur. This paper was presented at the Joint Highway Research Project Session of the 45th Annual Purdue Road School.

The paper discusses the problem under study and the progress through February, 1959. This project is one of the cooperative projects which we have with the State Highway Department and the Bureau of Public Roads.

The paper will be published in the Proceedings of the 45th Annual Road School and is presented to the Board for the record.

Respectfully submitted,

H. L. Michael

H. L. Michael, Secretary

HLM:ac

Attachment

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Technical Paper

HYDRAULICS OF RIVER FLOW UNDER ARCH BRIDGES

-A PROGRESS REPORT-

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Purdue University
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May 14, 1959

HYDRAULICS OF RIVER FLOW UNDER ARCH BRIDGES
-A PROGRESS REPORT-

By: H. J. Owen, A. Sooky, S. T. Husain, Graduate Assistants, Joint Highway Research Project, and J. W. Delleur, Associate Professor of Hydraulic Engineering, Purdue University.

Introduction:

Bridges are an integral part of any highway system. The design of a particular bridge depends on many variables. A major consideration, often necessary, is the length of the bridge. In certain circumstances and locations the length may be dictated by factors other than cost. But, where necessary the designer must decide, "How far can the bridge approaches extend onto the flood plain?". The shorter and therefore less expensive the bridge, the less waterway area generally provided. Waterway area is an important problem.

Bridge approaches extending far onto the flood plain decrease waterway area and produce, during high water, a large constriction causing excessive backwater and possible damage to the structure as well as unnecessary flooding of upstream areas. In some cases the state may be held liable for damage to property caused by bridge backwater.

In the past, studies pertaining to backwater caused by constrictions have considered shapes of opening such as that provided by a straight deck bridge. The Bureau of Public Roads has prepared, in cooperation with the Colorado State University, a report, entitled "Computation of Backwater Caused by Bridges."[1] This report in particular considered openings such as are provided by a straight deck bridge.

Figure 1 is a definition sketch for a normal crossing of the type used in the Colorado experiments. The typical water surface profile is shown as a solid line in view A. The maximum backwater superelevation is designated as h_1^* and the depression of the water surface below the normal

Superscript numbers refer to the bibliography at the end of the paper.

downstream of the constriction as h_3^*. View C of the figure illustrates the type opening studied.

Another research project whose subject is both interesting and applicable was carried out by New South Wales University of Technology.[2] This project was primarily concerned with sharp-edged rectangular openings which produced contraction ratios, m, from 0.10 to 0.95. An equation was presented which gives the discharge as a function of the width of opening, head over the opening including backwater, y_1, and C a coefficient depending on m and the Froude No. This equation makes it possible to determine the backwater if the discharge of the stream is known. Conversely, if the channel conditions are known the discharge through the opening may be calculated by a measurement of y_1.

Very little has been done with arched openings. In short, to the knowledge of the authors, no systematic study has been made of the hydraulics of flow under arch bridges. The arch bridge is unique in that the available waterway area decreases as the depth increases.

Figure 2 is a picture taken of the 10th Street bridge over Eagle Creek during the flood occurring in the summer of 1957 at Speedway, Indiana. The daming effect of the arch and the accumulated debris is visible. A typical multispan arch bridge subjected to flood flows is shown in Figure 3. This bridge is the Wayne Street Bridge over the Wabash River at Peru, Indiana.*

A project was initiated in the Hydraulics Laboratory at Purdue University to study this problem. It is sponsored by the State Highway Department of Indiana in cooperation with the U. S. Bureau of Public Roads.

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* Photographs courtesy of the Indiana State Flood Control and Water Resources Commission, Indianapolis.

Purpose:

The purpose, then, of the project is to:

1. Study the backwater produced by arches and develop a method for their computation.

2. Develop criterion for designing the proper clear span.

3. Study the hydraulic characteristics of flow under arch bridges including:

    a) single span bridges

    b) multiple span bridges

    c) various pier and abutment shapes

    d) shape of arch intrados

    e) discharge and slope of stream

    f) width of bridge.

This paper reports on the first year's work on this project. During this time, a preliminary investigation was initiated. Its purpose was, first to help in the design of the testing flume, and second to help in the design of the experiments to be carried out in the flume. Simultaneously, the design of the required testing facilities was done and the construction of these facilities was started.

Scope of Preliminary Experiments:

A dimensional analysis was made to define the important parameters of the flow. Originally, ten variables were considered as possibly of importance. Through the analysis the important parameters were found to be the Froude Number, the roughness, the contraction ratio and the normal depth of the flow. The dimensional analysis is presented in Appendix I.

For the purpose of the preliminary testing, a small variable slope

flume 6" wide and 12' long was built. Figure 4 shows the laboratory equipment used in the preliminary testing. To the right of the figure, the forebay is visible. The channel sides and bottom were constructed of lucite and carefully aligned by means of adjusting screws. The slope of the flume was controlled by a jack at the lower end of the flume. An I beam mounted horizontally above the flume served as a track for the mechanical and electrical point gages used in obtaining the water surface measurements.

An idealized two-dimensional case was investigated by using semicircular weirs with diameter along the bottom as shown in Figure 5 to represent the arch constriction. Sections B and C of Figure 5 illustrate the two types of surface profiles obtained with mild and steep slopes. An equation relating the depths upstream of the weir and the discharge was developed in two ways. The exact solution presents Q in terms of elliptic integrals:

$$Q = C_d \frac{4}{15}\sqrt{2g}\, b^{5/2}\left\{2\left(1-k^2+k^4\right)\left[E-E(\phi,k)\right]-\left(1-k^2\right)\left(2-k^2\right)\left[K-F(\phi,r)\right] - r^2\sin\phi\cos\phi\,\Delta\phi\left(3k^2\sin^2\phi-1-k^2\right)\right\} \tag{1}$$

Where K and E are the complete elliptic integrals of the first and second kind respectively, $F(\phi, k)$ and $E(\phi, k)$ are the incomplete elliptic integrals of the first and second kind, and

$$k = \sqrt{\frac{y_1+r}{b}} < 1 \quad , \qquad \phi = \sin^{-1}\sqrt{\frac{r}{y_1+r}}$$

$$\Delta\phi = \sqrt{1-k^2\sin^2\phi} = \sqrt{0.5}$$

An approximate solution gives Q in terms of an infinite series of powers of the ratio depth to radius:

$$Q = c\,y_1^{3/2}\, b A \qquad \text{where} \qquad c = C_d\frac{17}{24}\sqrt{2g}$$

$$\text{and} \quad A = \left[1-0.1294\left(\frac{y_1}{r}\right)^2 - 0.0177\left(\frac{y_1}{r}\right)^4 \;.\;.\;.\;.\;.\right] \tag{2}$$

The derivations of these formulas are given in Appendixes II and III.

The results of the weir tests were put in graphical form by plotting the coefficient of discharge vs the Froude Number with the contraction ratio as the parameter. The contraction ratio is defined as the ratio of the weir diameter b to the flume width B. This graph is shown in the upper left corner of Figure 6. The lower graph shows the relation of the Froude Number and the ratio of depth upstream of the weir to the normal depth.

The two-dimensional case was extended to the actual three-dimensional case by using semi-circular arch bridge models of the same contraction ratios. The bridge models were made of lucite--a typical water surface profile observation is shown in Figure 7. In that case, the flume walls were lined with copper wire mesh of 16 meshes per inch. This gave a Manning's roughness coefficient of approximately 0.025, which is typical of many canals and natural streams. The general test results of the extension of the theory to the three-dimensional case obtained in the smooth lucite channel and are shown in Figures 8 and 9.

Figure 8 shows the results of the three-dimensional tests, using bridge models of width L = 24 inches. The coefficients of discharge Cd and the ratio of the backwater depth y_1 to the normal depth y_o are plotted vs the Froude Number for several values of the contraction ratio m. The results of the two- and three-dimensional tests are compared in Figure 9. It is interesting to notice that for small Froude Numbers, say less than 0.5, Cd and the ratios $\frac{y_1}{y_o}$ are approximately the same for the two cases. For higher Froude Numbers, the three-dimensional tests exhibit smaller values of Cd and larger values of $\frac{y_1}{y_o}$.

Experimental Equipment and Test Plans:

Simultaneously with the preliminary testing the large flume was designed and is now under construction. The large flume will be 5 ft. wide, 64 ft. long and capable of 2 ft. maximum water depth. The structure is supported on six screw jacks with proportional rates of rise according to their positions. They will be driven by a common motor. These jacks will permit rapid and accurate changes of slope. Provision has been made for widening the channel eventually to 8 ft. Measurements of the water surface will be made from adjustable, stainless steel, guide rails running the length of the flume. The point gage to be used will be an electric indicating gage reading to a tenth of a milimeter. The flume will be provided with a tailgate control and a discharge control. Measurements of the discharge will be made with two venturi tubes.

The models tested will first be confined to single spans with no skew. Later tests will include other variables. Assuming a typical bridge cross section the scale ratio between model and prototype will probably be from 1:6 to 1:15 for single span bridges.

Figure 10 shows the arrangement of the equipment in the laboratory and a schematic drawing of the flume. The flume, as shown in the cross section, essentially consists of two I beams for longitudinal support, transverse 6" channels and rigidly attached vertical members. The spacing between the channel members is 2 ft. Adjustment bolts are provided for leveling and alignment of the inner channel. The inner channel will be $\frac{1}{4}$" plate steel.

This concludes the work done to date. This is a progress report on the first year of a three-year program.

The expected results in the future include data to compute back-water superelevation for different types of bridges and data to obtain the required waterway opening. The data would be given in the form of design curves for single or multispan bridges and for several shapes of piers and abutments.

APPENDIX I

Dimensional Analysis[3]

The Buckingham Theorem[4] states that in a physical problem including n quantities in which there are m dimensions, the quantities may be arranged into (n-m) dimensionless parameters. Suppose a dependent variable X_1 depends on the independent variables X_2, X_3 .....X_n

i.e., $$X_1 = f(X_2, X_3 \ldots\ldots X_n) \tag{3}$$

or $$g(X_1, X_2 \ldots\ldots\ldots X_n) = 0 \tag{4}$$

If π_1, π_2 etc. represent dimensionless grouping of the quantities X_1, X_2, X_3, etc. with m dimensions involved, equation (4) may be replaced by an equation of the form

$$h(\pi_1, \pi_2 \ldots\ldots\ldots \pi_{n-m}) = 0$$

The method of obtaining the π parameters is to select m of the X quantities, with different dimensions and that contain among them the m dimensions. These m quantities are used as basic variables together with one of the remaining n-m quantities for each π. For example let X_1, X_2, X_3 contain M, L and T, not necessarily in each one, but collectively.

The first parameter is $$\pi_1 = X_1^{a_1} X_2^{b_1} X_3^{c_1} X_4$$

The second one $$\pi_2 = X_1^{a_2} X_2^{b_2} X_3^{c_2} X_5$$

and so on until $$\pi_{n-m} = X_1^{a_{n-m}} X_2^{b_{n-m}} X_3^{c_{n-m}} X_m$$

In these equations the exponents are determined so that each is dimensionless. The dimensions of the X quantities are substituted and the exponents M, L, T are set equal to zero respectively. There will be three equations with three unknowns for each π parameter, so that the exponents a, b, c can be determined and hence the π parameters.

In the problem at hand, it is desired to determine the backwater superelevation caused by an arch bridge constriction. It is assumed that the variables which govern this backwater superelevation may be grouped into two categories: those describing the flow of the stream and those describing the bridge constriction. With reference to Figure 5, the following variables are considered:

a) For the stream flow

y_1, maximum water elevation upstream of constriction

y_0, the normal depth of flow in the approach channel

V_0, the velocity of flow at normal depth in the approach channel

n , Manning's roughness coefficient of the approach channel

ν , the kinematic viscosity

ρ , density of the fluid

g , the acceleration of gravity

Δh, the maximum drop of the water surface caused by the construction

b) For the structure--in order to simplify the problem let us consider at first single span circular arches with center on the bottom of the stream. The shape of the structure is thus fully determined by the diameter of the arch. The amount of constriction will also depend upon the initial width of the stream and the stage of the stream. The variables involved are thus:

B , width of stream at the bridge site

b , bridge opening at the spring line

y_0, normal depth.

Hence, from the above list of variables involved in our experiment:

$$y_1 = f(y_0, V, B, b, n, \mu, \rho, g, \Delta h) \tag{5}$$

or

$$g(y_1, y_0, V, B, b, n, \mu, \rho, g, \Delta h) = 0 \qquad (6)$$

There are m = 10 variables.

As n = 3 dimensions are involved, M, L, T, are the three dimensions selected, and there are m-n = 10 - 3 = 7 factors. The basic variables selected are V, ρ, y_o, which are helpful for complex situations. The seven parameters are:

$$\pi_1 = V^a \rho^b y_o^c g$$
$$\pi_2 = V^a \rho^b y_o^c \mu$$
$$\pi_3 = V^a \rho^b y_o^c n$$
$$\pi_4 = V^a \rho^b y_o^c y_1$$
$$\pi_5 = V^a \rho^b y_o^c b$$
$$\pi_6 = V^a \rho^b y_o^c \Delta h$$
$$\pi_7 = V^a \rho^b y_o^c B$$

by expanding the π quantities into dimensions

$$\pi_1 = (LT^{-1})^a (ML^{-3})^b (L)^c LT^{-2}$$

For L: $a - 3b + c\ 1 = 0$

For T: $-a - 2 = 0$

For M: $b = 0$

Solving: $a = -2,\ b = 0,\ c = 1$

Whence: $\pi_1 = \dfrac{y_o g}{V^2}$

Similarly: $\pi_2 = (LT^{-1})^a (ML^{-3})^b (L)^c ML^{-1} T^{-1}$

For L : $a - 3b + c = 1$

For T: $-a - 1 = 0$

For M: $b + 1 = 0$

Solving: $a = -1$, $b = -1$ and $c = -1$

Thus: $\Pi_2 = \frac{\mu}{V y_o}$

Also: $\Pi_3 = (LT^{-1})^a (ML^{-3})^b L^c L^{1/6} = 0$

For L: $a - 3b + c + 1/6 = 0$

For T: $-a = 0$

For M: $b = 0$

Solving: $c = -1/6$

Then: $\Pi_3 = \frac{n}{y_o^{1/6}}$

Also: $\Pi_4 = (LT^{-1})^a (ML^{-3})^b L^c L$

For L: $a - 3b + c + 1 = 0$

For M: $-a = 0$

For T: $+b = 0$

Solving: $c = -1$

Whence: $\Pi_4 = \frac{y_1}{y_o}$

Since B, b, h are linear dimensions similar to y_1:

$$\Pi_5 = \frac{b}{y_o}, \quad \Pi_6 = \frac{h}{y_o}, \quad \Pi_7 = \frac{B}{y_o}.$$

Introducing the Π factors, equation (6) may be replaced by

$$h_1\left(\frac{y_o g}{V^2}, \frac{\mu}{V \rho y_o}, \frac{n}{y_o^{1/6}}, \frac{y_1}{y_o}, \frac{B}{y_o}, \frac{b}{y_o}, \frac{\Delta h}{y_o}\right) = 0 \tag{7}$$

Inverting the first two parameters

$$h_2\left(\frac{V^2}{y_o g}, \frac{V \rho y_o}{\mu}, \frac{n}{y_o^{1/6}}, \frac{y_1}{y_o}, \frac{B}{y_o}, \frac{b}{y_o}, \frac{\Delta h}{y_o}\right) = 0$$

or

$$h_2\left(\frac{y_1}{y_o}, F_r^2, N_r, \frac{y_o^{1/6}}{n}, \frac{y_o}{B}, \frac{y_o}{b}, \frac{y_o}{\Delta h}\right) = 0 \tag{8}$$

Where F_r is the Froude Number $\frac{V}{\sqrt{g y_o}}$ and N_r is the Reynolds number $\frac{V y}{\nu}$. It is well known that gravity forces are predominant in open

channel flow whereas viscous forces play a secondary role. The Reynolds number may therefore be disregarded. Furthermore, assuming that the shape of the water surface downstream does not affect materially the shape of the water surface upstream, the term $\frac{y_0}{\Delta h}$ is also disregarded. Combining the ratios $\frac{b}{y_0}$ and $\frac{B}{y_0}$ into $\frac{b}{B}$ equation (8) may be replaced by:

$$\frac{y_1}{y_0} = \phi\left[F_r^2, \frac{y_0^{1/6}}{n}, \frac{b}{B}\right] \tag{9}$$

It is thus seen that the backwater superelevation is expected to be a function of the Froude Number, the roughness, the contraction ratio and the normal depth of the flow.

APPENDIX II

Derivations of Equations Governing the Flow in Rectangular Channels with Semi-Circular Constriction Plates:

The equation for the discharge in rectangular channels with a sharp crested semi-circular constriction is obtained and is expressed in terms of an infinite series of powers of the ratio $\frac{y_1}{r}$.

With reference to Figure 11, Bernoulli theorem gives

$$V = c\sqrt{2gh'} = c\sqrt{2g(y_1 - h)}$$

The element of area is $$dA = 2\sqrt{r^2 - h^2}\,dh$$

and the discharge is thus

$$Q = \int V\,dA = \int_0^{y_1} c\sqrt{2g(y_1 - h)} \cdot 2\sqrt{r^2 - h^2}\,dh \qquad (10)$$

Expanding into a series and integrating term by term and making use of the fact that $2r = b$:

$$Q = C_d\sqrt{2g}\,\frac{17}{24}\,y_1^{3/2}\,b\left[1 - 0.1294\left(\frac{y_1}{r}\right)^2 - 0.0177\left(\frac{y_1}{r}\right)^4 + \cdots\right] \qquad (11)$$

This may be written as

$$Q = C\,y_1^{3/2}\,b\,A \qquad (12)$$

where

$$C = C_d\,\frac{17}{24}\sqrt{2g} \qquad (13)$$

and

$$A = \left[1 - 0.1294\left(\frac{y_1}{r}\right)^2 - 0.0177\left(\frac{y_1}{r}\right)^4 + \cdots\right]. \qquad (14)$$

The discharge in the rectangular flume is given by

$$Q = V_0 A_0 = F_0\sqrt{g}\,B\,y_0^{3/2} \qquad (15)$$

where

$$F_0 = \frac{V_0}{\sqrt{g y_0}} \qquad (16)$$

is the Froude Number of the undisturbed flow.

Equating (11) and (15) and solving for the coefficient of discharge

$$Cd = \frac{12\sqrt{2}}{17} \cdot \frac{F_o}{m\,A} \left(\frac{y_o}{y_1}\right)^{3/2} \tag{17}$$

where

$$m = \frac{b}{B} \tag{18}$$

or

$$\frac{y_1}{y_o} = \left[\frac{12\sqrt{2}}{17} \cdot \frac{F_o}{m\,A\,Cd}\right]^{2/3}$$

Experimental values of Cd for Froude Numbers up to 1.5 and for three values of the contraction ratio ($m = 0.840$, $m = 0.672$, $m = 0.504$) are shown in Figure 6. From observed values of the discharge Q, and the water depths y_o and y_1, the values of Cd were computed from equation (17). Experimental values of $\frac{y_1}{y_o}$ are also presented in Figure 6 for the same ranges of the Froude Number and same values of the contraction ratio.

APPENDIX III

Transformation to Elliptic Integrals:

The integral of equation (10) may be evaluated in terms of complete and incomplete integrals of the first and second kind[5,6]. From Appendix II, equation (10), the theoretical discharge Q_t is obtained making the coefficient of discharge Cd equal to unity:

$$Q_t = 2\sqrt{2g}\int_0^{y_1}\sqrt{(y_1-h)(r^2-h^2)}\,dh \tag{19}$$

Let $\quad k^2 = \dfrac{y_1+r}{2r} \quad$ or $\quad y_1 = 2r(k^2-1),\ k<1,\ y_1<r \quad (20)$

and $$\mathrm{sn}^2u = \frac{h+r}{y_1+r} \tag{21}$$

Since $\quad \mathrm{sn}^2u + \mathrm{cn}^2u = 1$

then $$h = y_1\,\mathrm{sn}^2u - r\,\mathrm{cn}^2u = 2rk^2\sin^2u - r \tag{22}$$

and $$\frac{dh}{du} = 4.rk^2\,\mathrm{sn}u\,\frac{d}{du}\mathrm{sn}u = 4rk^2\,\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u \tag{23}$$

since $\quad \dfrac{d}{du}\left(\mathrm{sn}u\right) = \mathrm{cn}u\,\mathrm{dn}u$

From (21) and making use of (20)

$$\begin{aligned} y_1 - h &= y_1(1-\mathrm{sn}^2u) + r\,\mathrm{cn}^2u \\ &= y_1\,\mathrm{cn}^2u + r^2\mathrm{cn}^2u \\ &= 2rk^2\,\mathrm{cn}^2u \end{aligned} \tag{24}$$

Also from (21)

$$\begin{aligned} r^2 - h^2 &= 4r^2k^2\,\mathrm{sn}^2u\,(1-k^2\mathrm{sn}^2u) \\ &= 4r^2k^2\,\mathrm{sn}^2u\,\mathrm{dn}^2u \end{aligned} \tag{25}$$

since $\quad \mathrm{dn}^2u + k^2\mathrm{sn}^2u = 1$

Substituting (23), (24) and (25) into (19) the expression for the theoretical discharge becomes

$$\begin{aligned} Q_t &= 2\sqrt{2g}\int_{u_1}^{u_2}\left[2rk^2\mathrm{cn}^2u \cdot 4r^2k^2\mathrm{sn}^2u\,\mathrm{dn}^2u\right]^{1/2} 4rk^2\,\mathrm{sn}u\,\mathrm{dn}u\,\mathrm{cn}u\,du \\ &= 32\sqrt{g}\,r^{5/2}k^4\int_{u_1}^{u_2}\mathrm{cn}^2u.\mathrm{sn}^2u\,\mathrm{dn}^2u\,du \end{aligned} \tag{26}$$

The lower limit u_1 is obtained from (21) as follows:

$$sn^2 u_1 = \frac{0+r}{y_1+r}$$

$$\text{or} \quad sn\, u_1 = \sin\phi = \sqrt{\frac{r}{y_1+r}}$$

where $\phi = am\ u_1$

and finally $u_1 = F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin\phi}}$, $k < 1$ (27)

$F(\phi, k)$ is the incomplete elliptic integral of the first kind

The upper limit u_2, is obtained from (21) as follows:

$$sn^2 u_2 = \frac{y_1+r}{y_1+r} = 1$$

or $\quad sn\, u_2 = 1$

and

$$u_2 = K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

where K is the complete elliptic integral of the first kind (28)

The expression for the theoretical discharge (26) becomes

$$Q_t = 32\sqrt{g}\, r^{5/2} k^4 \int_{F(\phi,k)}^{K} cn^3 u\, dn^2 u\, sn^2 u\, du \tag{29}$$

Upon performing the integration, and introducing the diameter $b = 2r$

$$Q_t = \frac{4}{15}\sqrt{2g}\, b^{5/2} \Big\{ 2(1-k^2+k^4)\big[E - E(\phi,k)\big] - (1-k^2)(2-k^2)\big[K - F(\phi,k)\big] - k^2 \sin\phi\cos\phi\,\Delta\phi\,(3k^2\sin^2\phi - 1 - k^2) \Big\} \tag{30}$$

where

$$E = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\phi}\, d\phi \tag{31}$$

which is the complete elliptic integral of the second kind, and

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\phi}\, d\phi \tag{32}$$

which is the incomplete elliptic integral of the second kind, and

$$\phi = \sin^{-1}\sqrt{\frac{r}{y_1+r}} \tag{33}$$

and

$$\Delta\phi = \sqrt{1-k^2\sin^2\phi} = \sqrt{0.5} \tag{34}$$

[illegible]

and finally

$$k = \sqrt{\frac{y_1 + r}{b}} \ . \tag{35}$$

Equation (30) yields the theoretical discharge for the flow through a semi-circular constriction of diameter $b = 2r$ and where the maximum depth upstream of the constriction is y_1. The quantities K, E, F (ϕ, k), E (ϕ, k) may be obtained from tables. Equation (30) is somewhat similar to that for the flow through circular weirs obtained by J. C. Stevens[7] which is

$$Q_t = \frac{4}{15}\sqrt{2g}\, D^{5/2}\left\{2(1-k^2+k^4)\,E - (2-k^2)(1-k^2)\,K\right\} \tag{36}$$

where $k^2 = H/D$, H being the head over the invert, and D is the diameter of the circular weir. Stevens also gives an infinite series approximation to equation (36) which is similar to equation (11):

$$Q_t = 2\sqrt{2g}\, D^{5/2}\,\pi\left(\frac{1}{8}z^2 - \frac{1}{32}z^3 - \frac{5}{1024}z^4 \ldots\ldots\right) \tag{37}$$

where $z = H/D$.

SYMBOLS

A	An infinite series of powers of the ratio depth to radius.
dA	Elementary area
B	Width of arch at spring line, or width of flume.
b	Diameter of the arch or weir.
c	Coefficient depending on the coefficient of discharge, equation (13).
Cd	Coefficient of discharge.
E	Complete elliptic integral of the second kind, equation (31).
$E(\phi, k)$	Incomplete elliptic integral of the second kind, equation (32).
F_r	Froude Number of the flow, equation (16).
$F(\phi, k)$	Incomplete elliptic integral of the second kind, equation (32).
g	Acceleration of gravity.
h	Hydraulic head.
h_1^*	Maximum backwater superelevation.
h_3^*	Depression of the water surface below the normal downstream of the constriction.
Δh	Drop in water surface caused by the constriction.
K	Complete elliptic integral of the first kind, equation (28).
k	Modulus of elliptic integrals, equation (20).
L	Width of bridge, measured along waterway axis.
m	Contraction ratio = b/B.
n	Manning's roughness factor.
N_r	Reynolds Number.

Q	Actual discharge.
Q_t	Theoretical discharge.
r	Radius of the arch.
sn u, cn u, dn u	Jacobi elliptic functions.
V_o	Velocity at normal depth in approach channel.
y_o	Normal depth in approach channel.
y_1	Maximum water depth upstream of constriction.
ν	Kinematic viscosity.
ρ	Density of the fluid.
ϕ	Amplitude in elliptic functions, ϕ = am u, equation (33).

ACKNOWLEDGMENT

The project is sponsored jointly by the Indiana State Highway Department and the Bureau of Public Roads and is administered by the Joint Highway Research Project at Purdue University. The research is under the direction of Dr. J. W. Delleur, Associate Professor of Hydraulic Engineering. Mr. H. J. Owen is responsible for the design of the main testing facilities and the supervision of its assembly. He presented the oral version of this paper at the 45th Road School. Mr. S. T. Husain designed and built the small testing flume, prepared the dimensional analysis of Appendix I and did some observations of the free surface profile. Mr. A. Sooky derived the equations governing the flow in rectangular channels with semicircular constrictions detailed in Appendixes II and III and carried out the corresponding experiments including the extension to the three-dimensional case. Mr. P. F. Biery was responsible for preparation of the plates, helped in the experiments and in the construction of both flumes.

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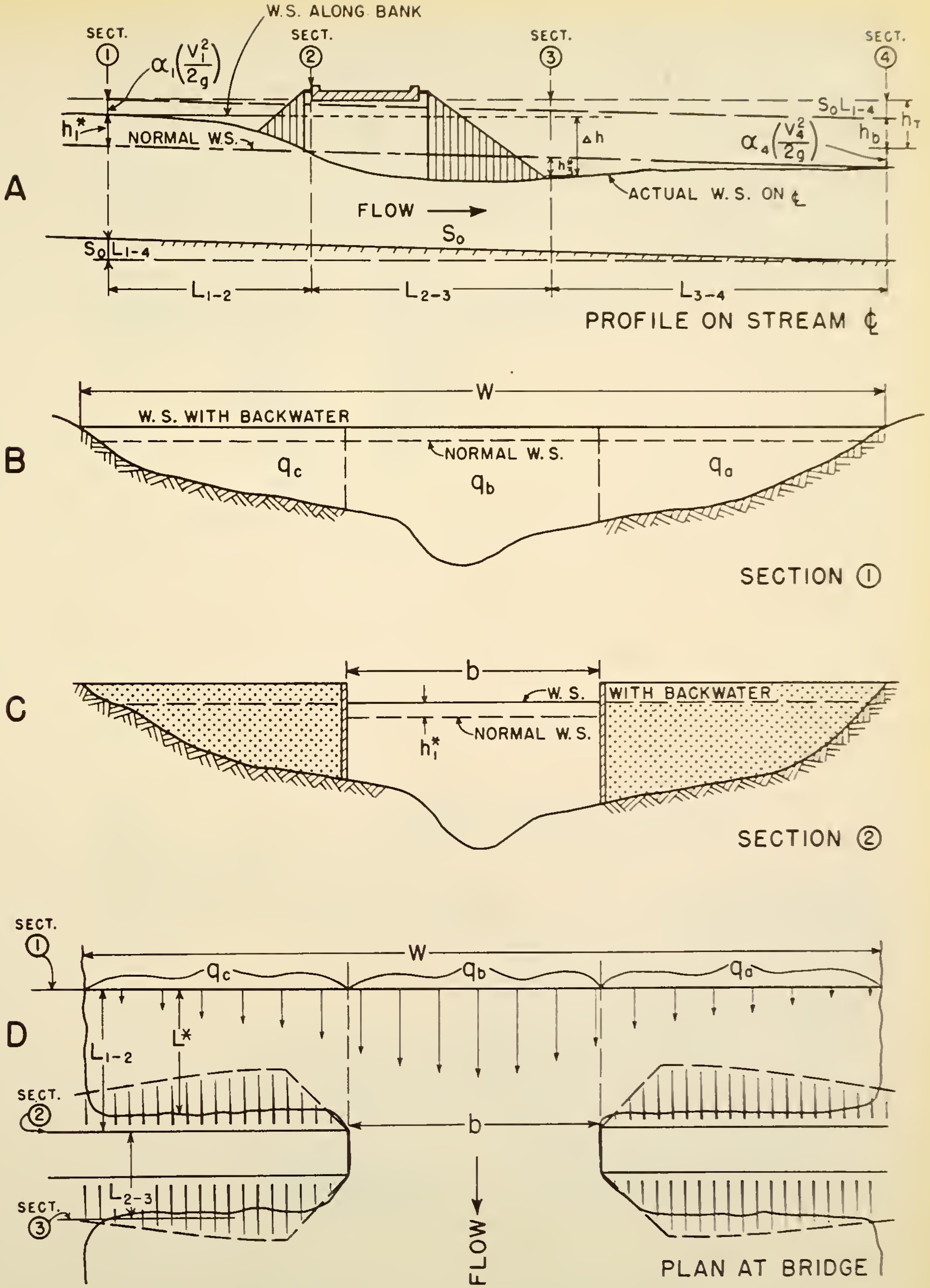
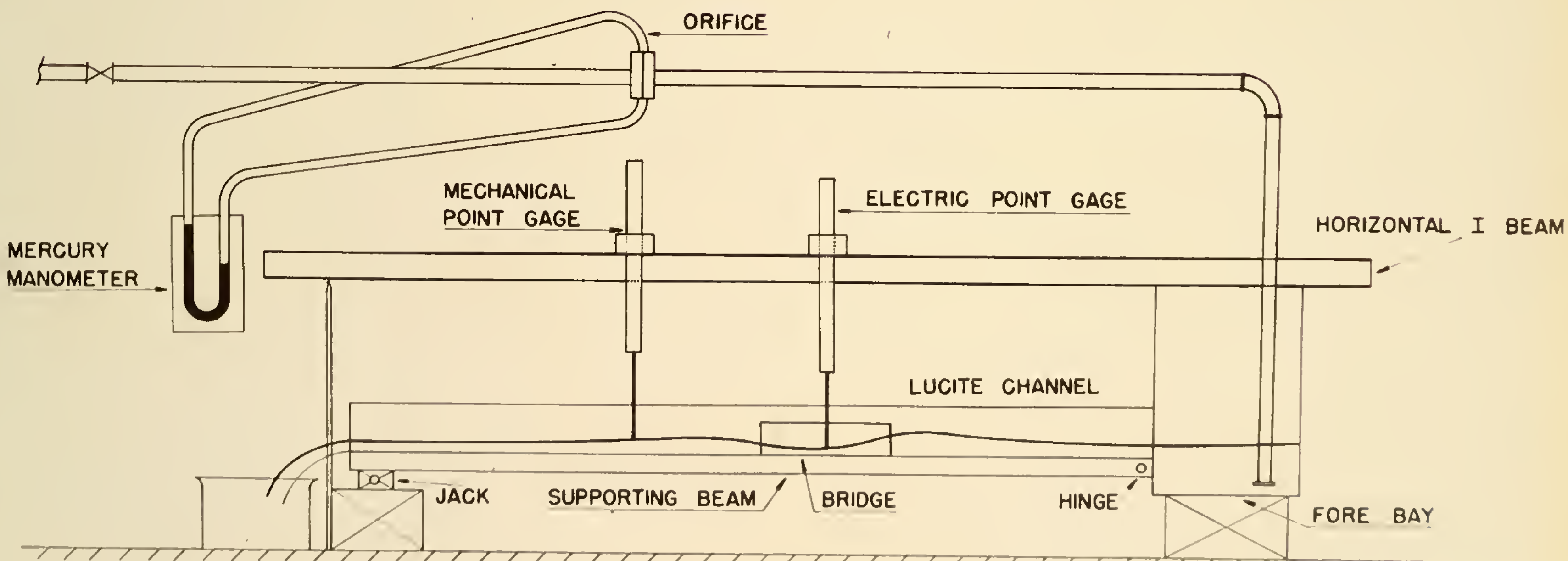


FIG. 1

NORMAL CROSSING
WINGWALL ABUTMENTS

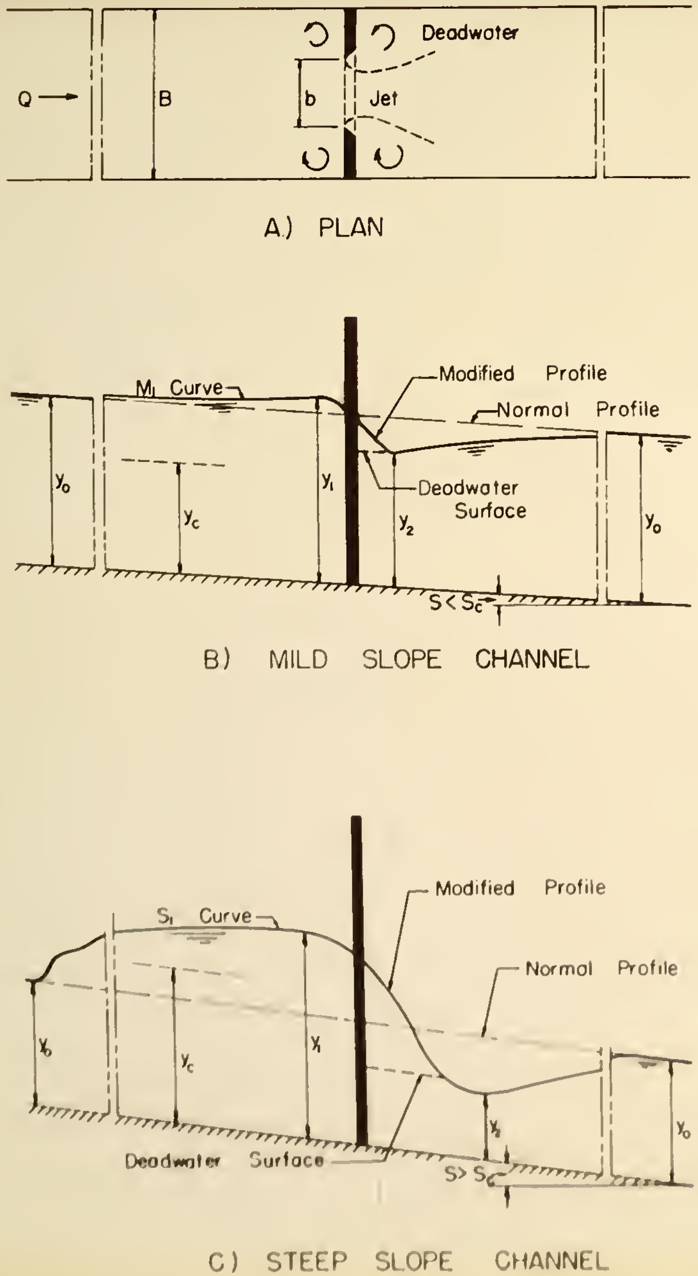
FIG. 2

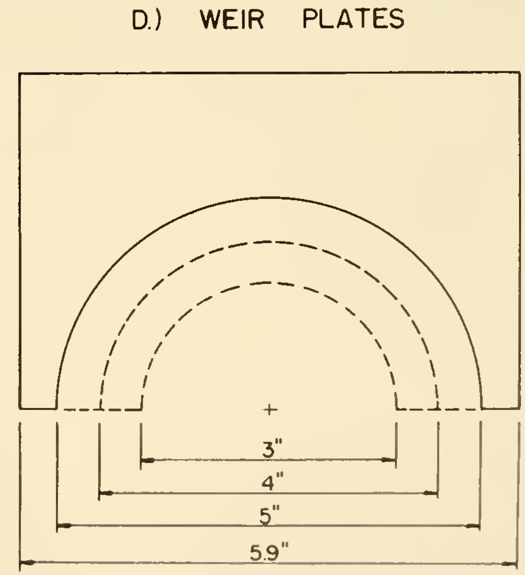
FIG. 3



PRELIMINARY FLUME

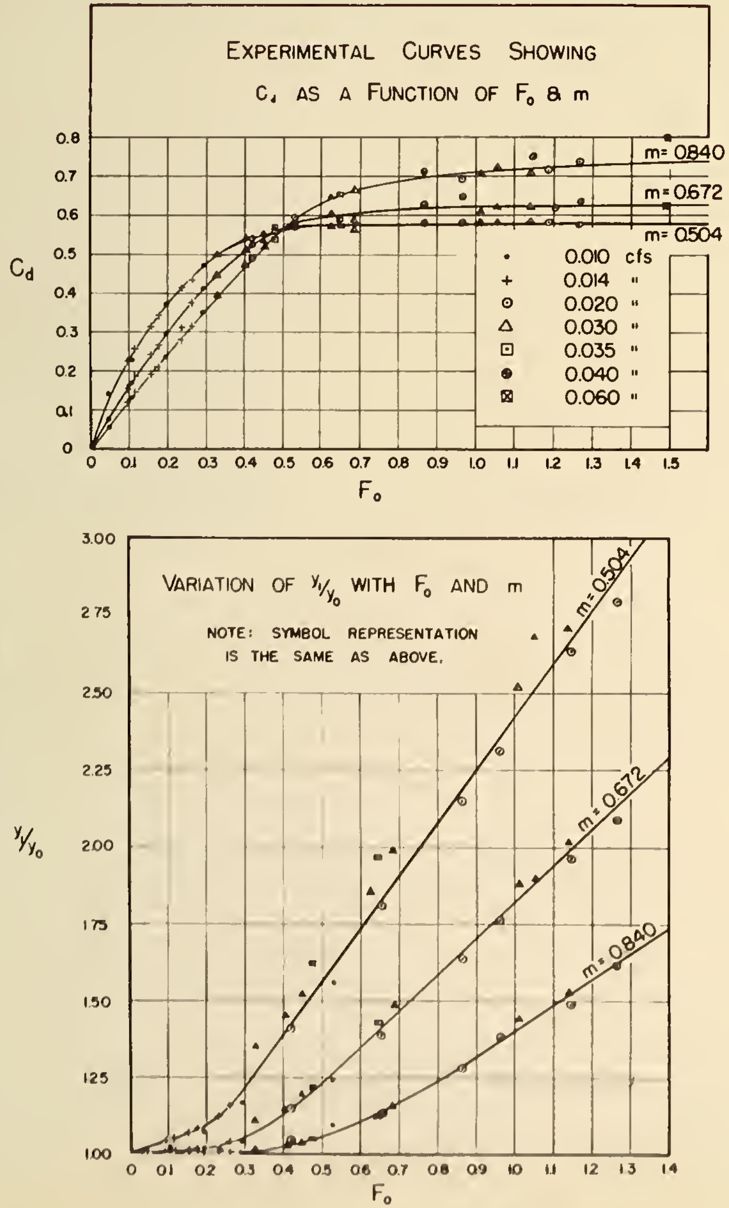
FIG. 4

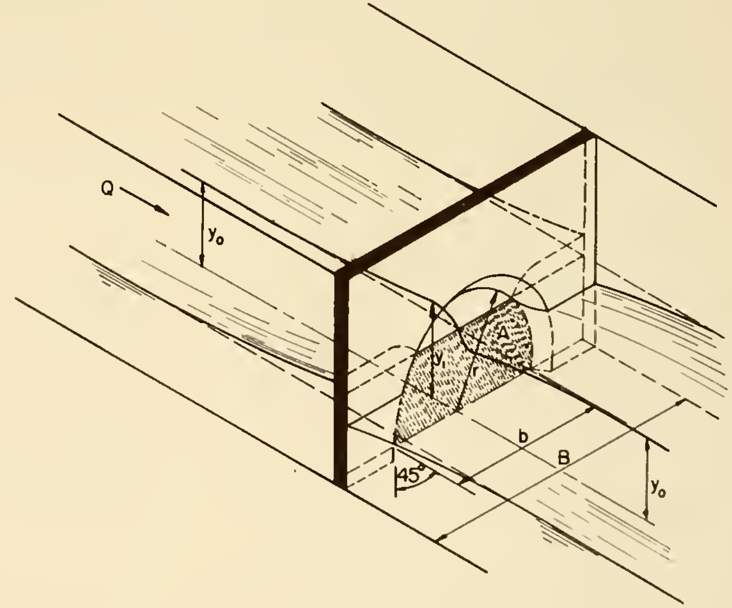




EFFECT OF CHANNEL CONSTRICTION ON WATER SURFACE PROFILE

FIG. 5





$$Q = C y_1^{3/2} b A \qquad (1)$$

$$C = c_d \sqrt{2g}\ 17/24$$

$$A = \left[1 - 0.1294 (y_1/r)^2 - 0.0177 (y_1/r)^4 + \cdots\right]$$

$$y_1/y_o = \left(F_o \sqrt{g} / C m A\right)^{2/3} \qquad (2)$$

$$F_o = V_o / \sqrt{g y_o}$$

$$m = b/B$$

FIG. 6

SEMICIRCULAR WEIR TESTS

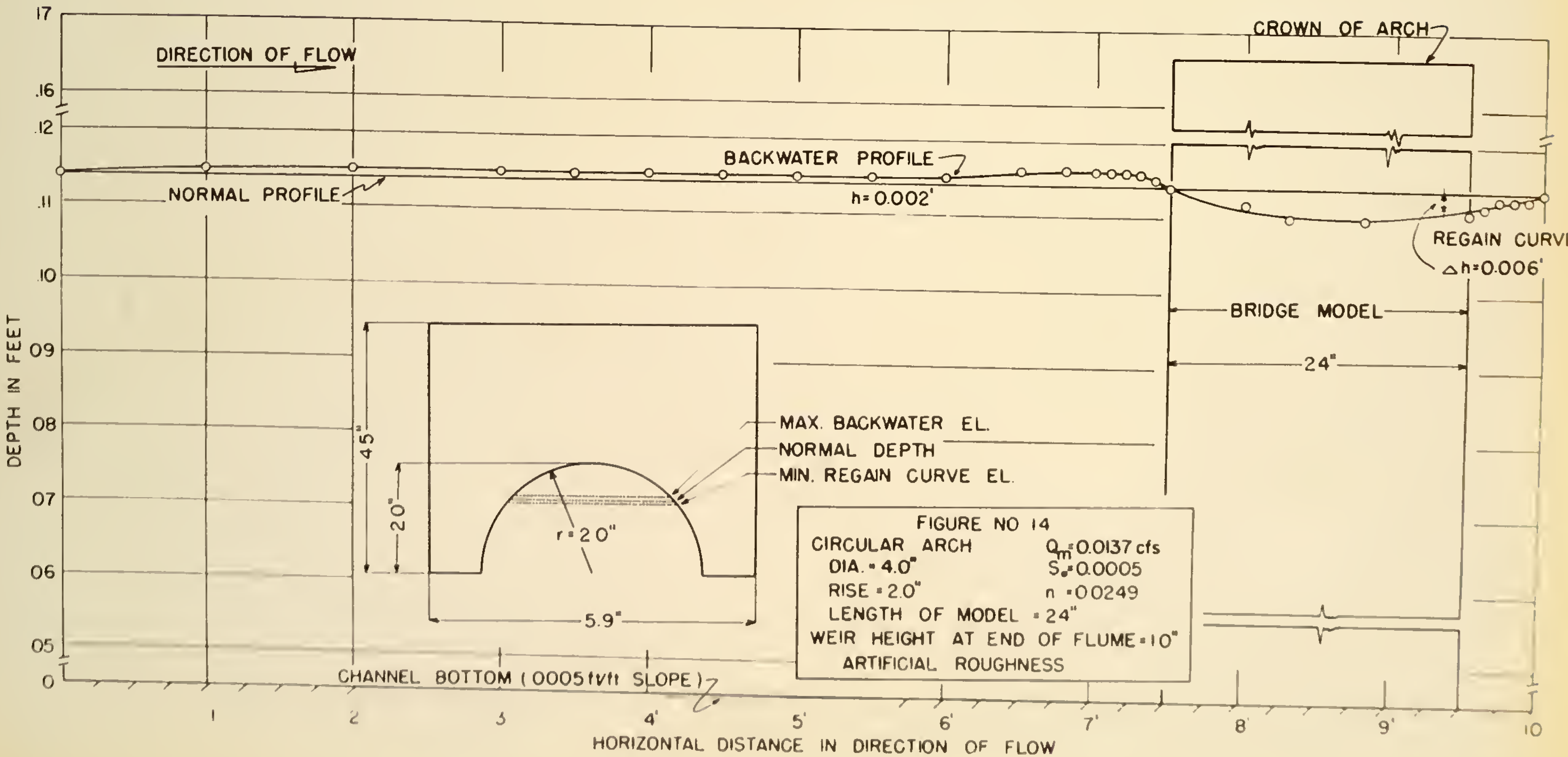
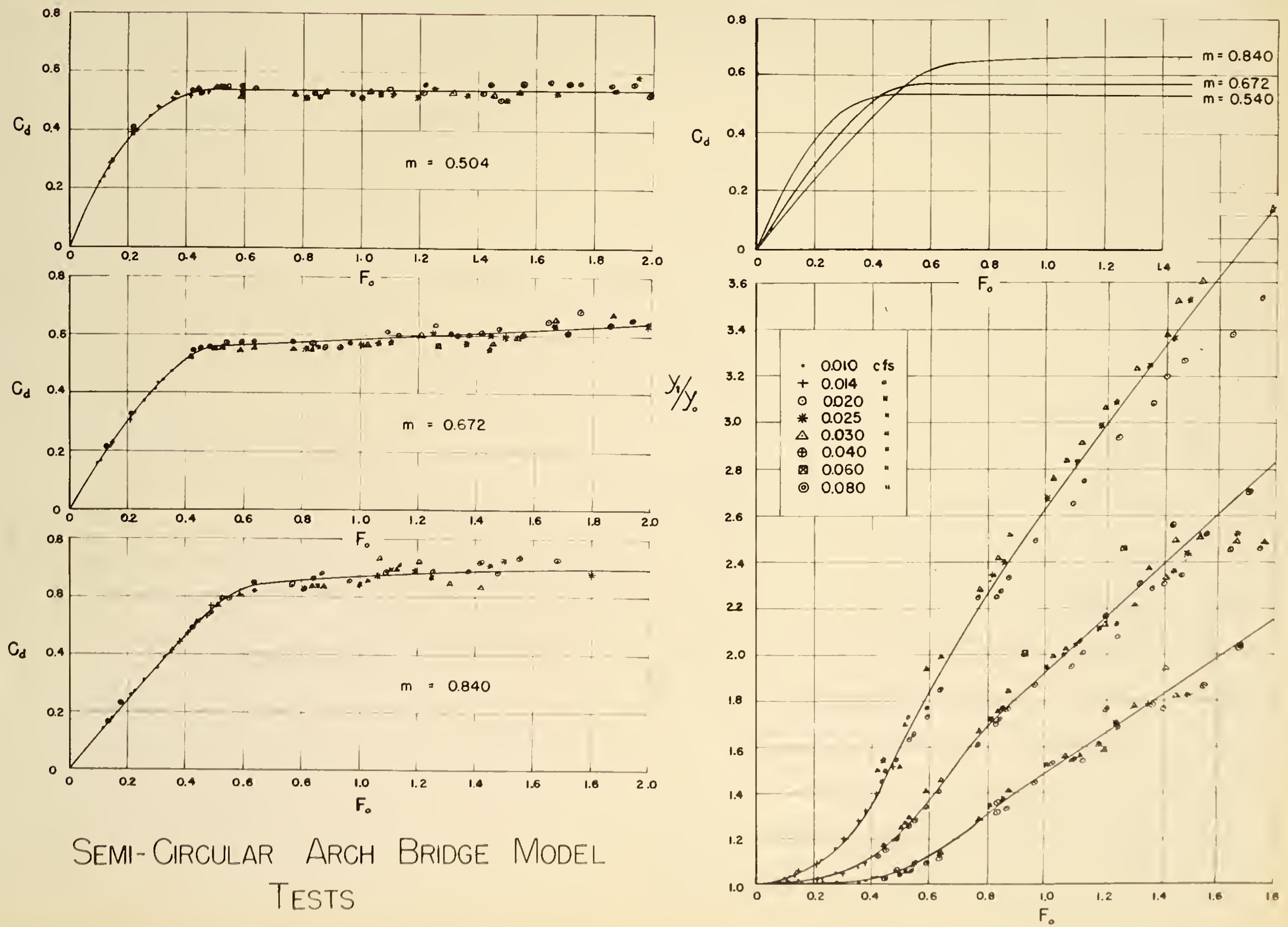
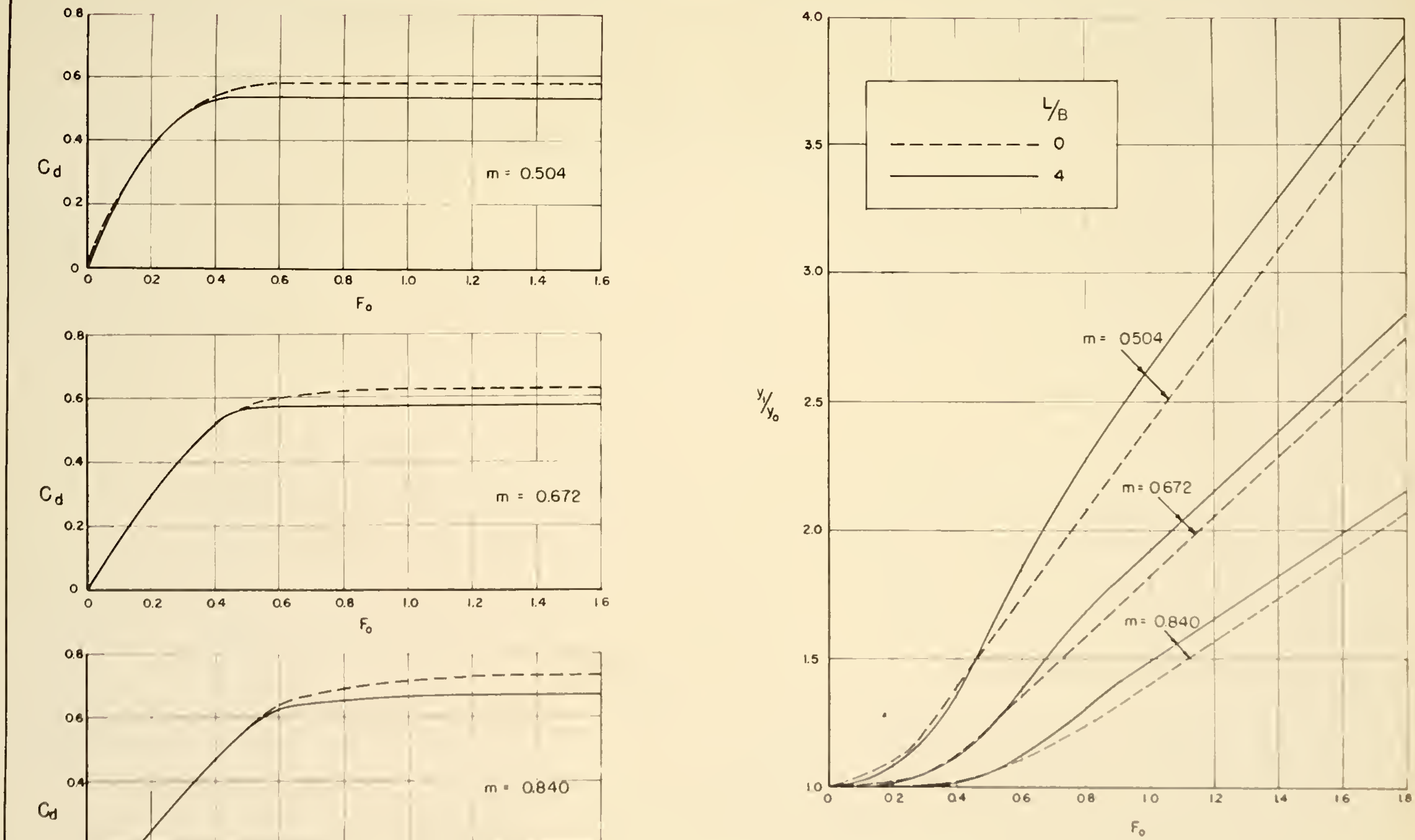

DIRECTION OF FLOW
CROWN OF ARCH
BACKWATER PROFILE
NORMAL PROFILE
h=0.002'
REGAIN CURVE
△h=0.006'
BRIDGE MODEL
24"
MAX. BACKWATER EL.
NORMAL DEPTH
MIN. REGAIN CURVE EL.
45"
20"
r=2.0"
5.9"
FIGURE NO 14
CIRCULAR ARCH
DIA.=4.0"
RISE=2.0"
LENGTH OF MODEL =24"
WEIR HEIGHT AT END OF FLUME=10"
ARTIFICIAL ROUGHNESS
Qm=0.0137 cfs
So=0.0005
n =0.0249
CHANNEL BOTTOM (.0005 ft/ft SLOPE)
DEPTH IN FEET
HORIZONTAL DISTANCE IN DIRECTION OF FLOW

FIG. 7



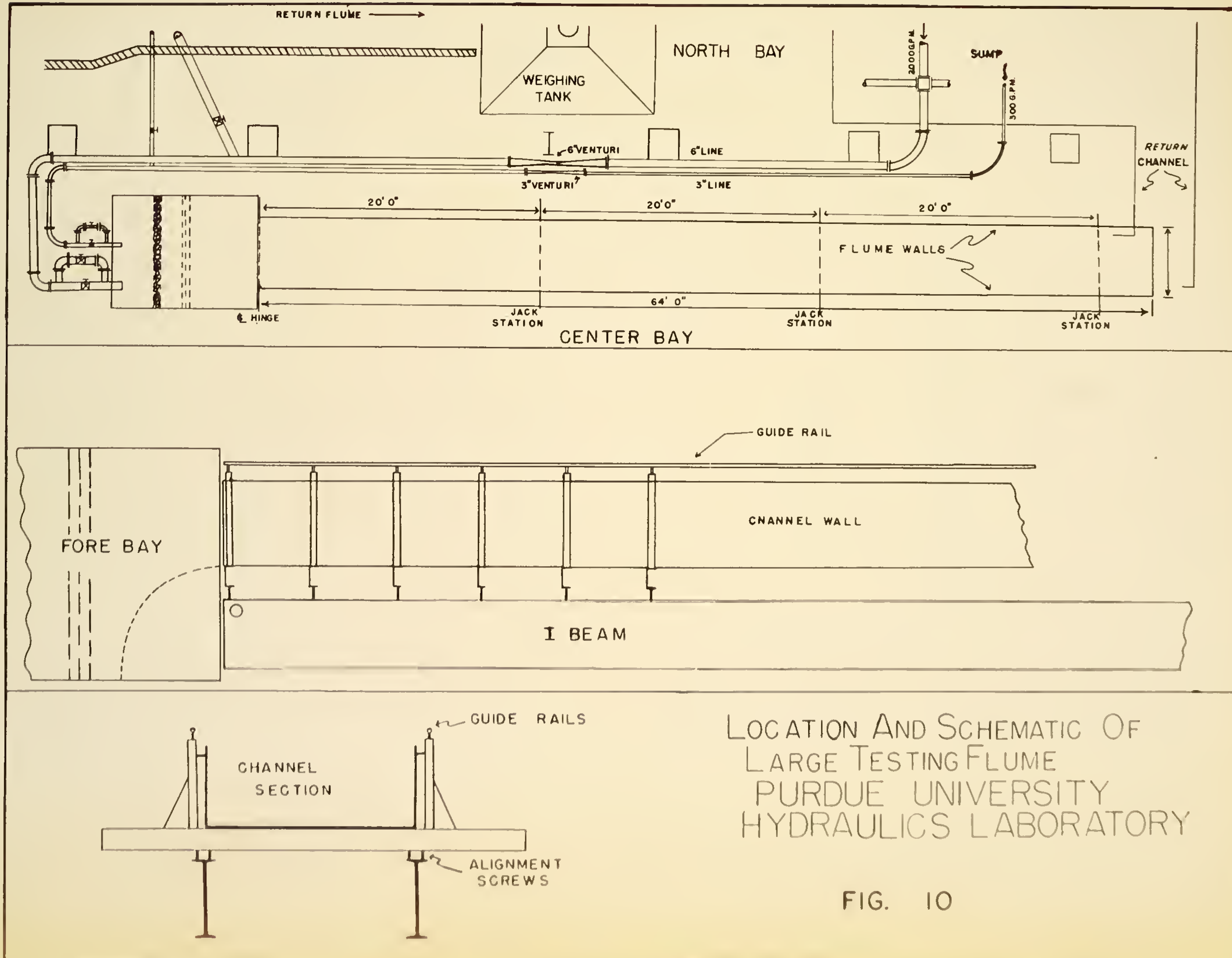
SEMI-CIRCULAR ARCH BRIDGE MODEL TESTS

FIG. 8



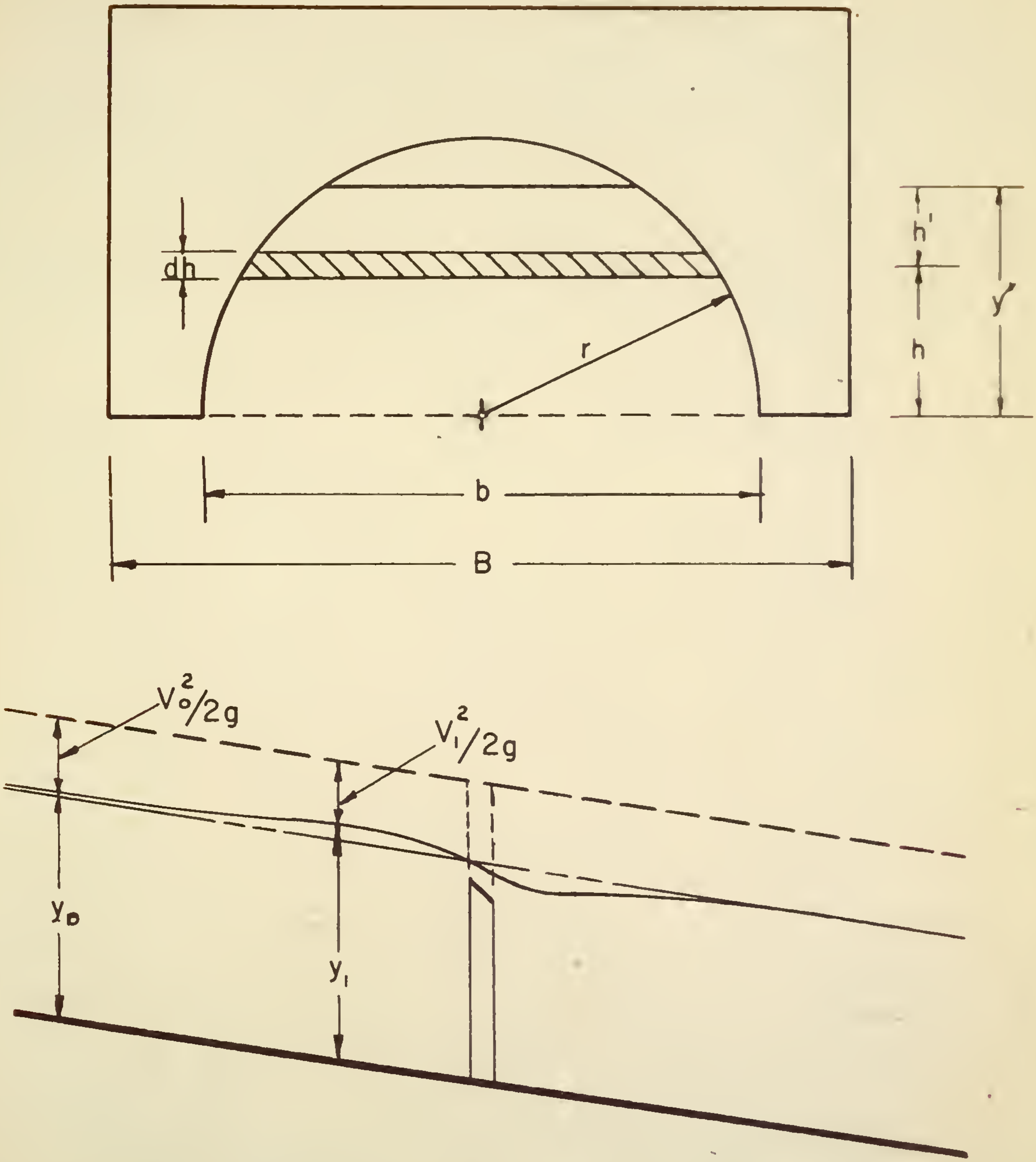
FLOW IN RECTANGULAR CHANNELS WITH SEMI-CIRCULAR CONSTRICTIONS - COMPARISON OF TWO AND THREE DIMENSIONAL CASES

FIG 9



LOCATION AND SCHEMATIC OF
LARGE TESTING FLUME
PURDUE UNIVERSITY
HYDRAULICS LABORATORY

FIG. 10



DEFINITION SKETCH

FIG. 11